Non-generic spectral statistics in the quantized stadium billiard

1993 J. Phys. A: Math. Gen. 26 6217

(http://iopscience.iop.org/0305-4470/26/22/022)

J. Phys. A: Math. Gen. 26 (1993) 6217–6230. Printed in the UK

Non-generic spectral statistics in the quantized stadium billiard

M Sieber†, U Smilansky†, S C Creagh‡ and R G Littlejohn§

† Department of Nuclear Physics, The Weizmann Institute of Science, Rehovot 76100, Israel
‡ Niels Bohr Institute, Blegdamsvej 17, Dk-2100 Copenhagen 0, Denmark
§ Lawrence Berkeley Laboratory and University of California, Berkeley, CA 94720, USA

Received 17 May 1993

Abstract. We consider the effect of a continuous family of neutral (bouncing ball) orbits on the energy spectrum of the quantized stadium billiard. Using a semiclassical approximation we derive analytic expressions for standard two-point spectral measures. The corrections due to the bouncing ball orbits account for some of the non-generic features observed in the analysis of the spectrum of a stadium cavity which was recently measured. Once the bouncing ball contributions are subtracted, the spectrum is shown to be well reproduced by the semi-classical trace formula based on unstable periodic orbits. We also study special patterns in the spectrum which are due to other non-generic features such as edge effects and 'whispering gallery' trajectories.

1. Introduction

The stadium billiard is one of the standard examples of a strongly chaotic system. It is one of the few cases which are proven to be K- and B-systems [1, 2], which display ergodicity and mixing. The stadium billiard was one of the first chaotic systems to be quantized, and where the levels statistics was shown to follow expressions derived from Wigner's and Dyson's random matrix theory (RMT) [3]. This observation marked the beginning of intensive research on the implications of classical chaos to the statistics of energy levels in the corresponding quantum system. The interest in the level statistics of the quantized stadium billiard was revived recently following a high-precision measurement of the first 1060 levels of a super-conducting stadium cavity [4]. A straightforward analysis of the spectral fluctuations showed a systematic deviation of the long-range correlations in the Δ_3-statistics from the RMT prediction. In this paper we will explain the reason for this behaviour and show in detail how the good agreement with RMT can be restored. The authors of [4] actually applied our method to their data.

The semiclassical theory which shows that for generic systems the level statistics are universal and follow the results of RMT was developed by Berry [5] on the basis of Gutzwiller's periodic orbit theory for the spectral density [6, 7]. A crucial assumption for the derivation of universality is that all the periodic orbits of the system are isolated and unstable. In the stadium billiard, however, there exists a non-generic set of neutral periodic orbits—the orbits which bounce perpendicularly between the parallel straight sections of the billiard boundary ('bouncing ball orbits'). Such a continuous set of neutral orbits gives an additional contribution to the semiclassical expression for the level density [8, 9]. Continuous sets of neutral orbits generically appear in integrable systems for which their semiclassical contribution to the level density has been derived in [10]. We show in this

0305-4470/93/226217+14$07.50

paper that the deviation of the energy statistics of the stadium billiard can be understood within the framework of Berry's theory, if one additionally includes the contributions of the family of 'bouncing ball orbits'. Our calculations can be extended in a straightforward manner to other strongly chaotic systems for which families of neutral periodic orbits exist.

We also analyse the Fourier transform of the spectral density. We show that the length spectrum can be approximated well in terms of the contribution of the family of 'bouncing ball trajectories' plus the contributions from periodic orbits according to Gutzwiller's theory plus some additional edge corrections. Numerical examinations are carried out using the measured energy spectrum of [4].

2. The semiclassical energy density

2.1. The contribution of the family of neutral periodic orbits

We consider a desymmetrized version of the stadium billiard which consists of a quarter circle of radius a adjoint to a rectangle with side lengths a and b (see figure 5). The propagator $K(q'', q', t)$ can be semiclassically approximated by a sum over all classical paths from q' to q''

$$K(q'', q', t) \approx \sum_{\text{cl.paths}} \frac{1}{2\pi i\hbar} \left| \det\left(-\frac{\partial^2 R}{\partial q'' \partial q'} \right) \right|^{1/2} \exp\left\{ \frac{i}{\hbar} R(q'', q', t) - i\frac{\pi}{2}\nu \right\} . \tag{1}$$

The classical action $R(q'', q', t)$ of a path from q' to q'' is given by

$$R(q'', q', t) = \int_{t'}^{t''} d\tau\, L(q, \dot{q}, \tau) \qquad t = t'' - t' \tag{2}$$

where $L(q, \dot{q}, t) = m\dot{q}^2/2$ denotes the Lagrangian of the system. The phase ν is the number of conjugate points along the classical path plus twice the number of reflections on the boundary.

The sum over paths is divided into two parts in order to obtain the contributions of the 'bouncing ball orbits'

$$K(q'', q', t) \approx K_b(q'', q', t) + K_r(q'', q', t) . \tag{3}$$

The first part $K_b(q'', q', t)$ contains only contributions from paths which are reflected on the two straight line segments of length b only, and the second part $K_r(q'', q', t)$ contains the contributions of all remaining paths. The classical paths which contribute to $K_b(q'', q', t)$ can be obtained by the principle of mirror images. Consider the infinite number of straight lines which are parallel to the straight line segment of length b and which all have a distance a to their neighbouring lines. Then there is a one-to-one correspondence between all the paths in the billiard from q' to q'' and all paths from q' to the set of points which are obtained by arbitrary reflections of q'' on the parallel straight lines. The lengths of these paths are given by

$$\begin{aligned} L_e(n) &= [(x_2 - x_1)^2 + (y_2 - y_1 + 2an)^2]^{1/2} \qquad n \in \mathbb{Z} \\ L_o(n) &= [(x_2 - x_1)^2 + (y_2 + y_1 + 2an)^2]^{1/2} \qquad n \in \mathbb{Z} . \end{aligned} \tag{4}$$

The lengths $L_e(n)$ correspond to paths with an even number of reflections on the boundary, whereas the lengths $L_o(n)$ correspond to paths with an odd number of reflections. One obtains

$$K_b(q'', q', t) = \frac{m}{2\pi i\hbar t} \sum_{n=-\infty}^{+\infty} \left[\exp\left\{ \frac{im}{2\hbar t} L_e^2(n) \right\} - \exp\left\{ \frac{im}{2\hbar t} L_o^2(n) \right\} \right] . \tag{5}$$

The outgoing Green function is related to the propagator by the following equation

$$G(q'', q', E) = \lim_{\varepsilon\to 0} \frac{1}{i\hbar} \int_0^\infty dt\, K(q'', q', t) \exp\left\{ \frac{i}{\hbar}(E + i\varepsilon)t \right\} . \tag{6}$$

After inserting (5) the integral can be done exactly [11] and results in

$$G_b(q'', q', E) = \frac{m}{2i\hbar^2} \sum_{n=-\infty}^{+\infty} \left[H_0^{(1)}\big(kL_e(n)\big) - H_0^{(1)}\big(kL_o(n)\big) \right] . \tag{7}$$

Here $H_0^{(1)}$ denotes a Hankel function and $k = \sqrt{2mE}/\hbar$. In the usual semiclassical approximation for the Green function the integral in (6) is done by a stationary phase approximation. The semiclassical result for the function $G_b(q'', q', t)$ can be obtained from (7), if one replaces the Hankel function by its asymptotic approximation $H_0^{(1)}(x) \sim \sqrt{2/(\pi x)} \exp\{ix - i\pi/4\}$, $x \to \infty$.

The contribution of the 'bouncing ball trajectories' to the semiclassical level density is obtained through the relation

$$d(E) = -\frac{1}{\pi} \operatorname{Im} \int dq\, G(q, q, E) . \tag{8}$$

It is convenient to express the level density in terms of the wavenumber k

$$\hat{d}(k) = \frac{\hbar^2 k}{m} d(E) . \tag{9}$$

One obtains

$$\begin{aligned} \hat{d}_b(k) &= \int_0^b dx \int_0^a dy\, \operatorname{Im} \left\{ \frac{ik}{2\pi} \sum_{n=-\infty}^{+\infty} \left[H_0^{(1)}(2ka|n|) - H_0^{(1)}(2k|y + an|) \right] \right\} \\ &= \frac{abk}{2\pi} \sum_{n=-\infty}^{+\infty} J_0\big(2ka|n|\big) - \frac{b}{2\pi} \end{aligned} \tag{10}$$

here the first term in the second line of (10) contains the contributions of the orbits of the even class, which are periodic orbits and their multiple traversals. The contributions of the orbits of the odd class, which are closed but non-periodic orbits, were summed up to give the second term in (10). Using the asymptotic approximation for the J_0 Bessel function one obtains the following approximation to $\hat{d}_b(k)$

$$\hat{d}_b(k) \approx \frac{abk}{2\pi} - \frac{b}{2\pi} + \frac{b}{\pi}\sqrt{\frac{ak}{\pi}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \cos(2ank - \pi/4) . \tag{11}$$

There are two different contributions to $\hat{d}_{\mathrm{b}}(k)$

$$\hat{d}_{\mathrm{b}}(k) \approx: \langle \hat{d}_{\mathrm{b}}(k) \rangle + \hat{d}_{\mathrm{b}}^{\mathrm{osc}}(k) \,. \tag{12}$$

$\hat{d}_{\mathrm{b}}^{\mathrm{osc}}(k)$ denotes the sum over oscillating terms, whereas $\langle \hat{d}_{\mathrm{b}}(k) \rangle$ denotes the first two terms on the right-hand side of (11).

In a billiard, the mean level density is asymptotically given by the generalized Weyl law [12]

$$\langle \hat{d}(k) \rangle \sim \frac{Ak}{2\pi} - \frac{L}{4\pi} + \cdots \qquad k \to \infty \,. \tag{13}$$

Here A and L are the area and the length of the boundary of the billiard system, respectively. Thus, $\langle \hat{d}_{\mathrm{b}}(k) \rangle$ consists of the contribution of the rectangle $a \times b$ to the leading term of the asymptotic approximation for $\langle \hat{d}(k) \rangle$, and of the contribution of the two straight sections of the boundary of length b to the next-to-leading term.

The fact that starting from (5) the steps can be done exactly without the stationary phase approximation enables us to see in this case the effect of the usual semiclassical approximation, which is used for example for the derivation of Gutzwiller's periodic-orbit theory. The contributions of the closed but non-periodic trajectories which are neglected in the stationary phase approximation yield the next-to-leading term in Weyl's formula. If one calculates this term after replacing the Hankel functions in (7) by their asymptotic approximation, one obtains an expression which is wrong by a factor of $\sqrt{2}$.

Finally, an alternative expression for $\hat{d}_{\mathrm{b}}(k)$ in which the k-dependence can be seen more explicitly can be obtained by transforming (10) with the use of the Poisson summation formula. This results in

$$\hat{d}_{\mathrm{b}}(k) = \frac{bk}{\pi} \sum_{M=1}^{\infty} (k^2 - k_M^2)^{-1/2} \Theta(k - k_M) \tag{14}$$

where Θ is the Heaviside theta function and $k_M = \pi M/a$ are the wavenumbers corresponding to the wavefunctions in a one-dimensional infinitely high square well of width a. Although the function $\hat{d}_{\mathrm{b}}(k)$ diverges at $k = k_M$, these wavenumbers do not necessarily correspond to semiclassical energies of the stadium billiard, since the order of the divergence is one half and not one. A discussion of such 'false' singularities for the case of the motion on a torus is given in [13].

From the results for the level density it is straightforward to obtain the semiclassical contributions to the spectral staircase $\hat{N}(k)$ which is defined as the number of eigenvalues of the stadium with wavenumber below k

$$\hat{N}(k) = \#\{k_n | k_n \leqslant k\} = \int_0^k \mathrm{d}k' \, \hat{d}(k') \,. \tag{15}$$

Integration of (10) yields

$$\begin{aligned} \hat{N}_{\mathrm{b}}(k) &= \frac{abk^2}{4\pi} - \frac{bk}{2\pi} + \frac{bk}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n} J_1(2kan) \\ &\approx \frac{abk^2}{4\pi} - \frac{bk}{2\pi} + \frac{b}{2\pi} \sqrt{\frac{k}{\pi a}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}} \cos(2kan - 3\pi/4) \,. \end{aligned} \tag{16}$$

Alternatively, the result of integrating (14) is given by

$$\hat{N}_b(k) = \frac{b}{\pi} \sum_{M=1}^{\infty} \sqrt{k^2 - k_M^2}\,\Theta(k - k_M)\,. \tag{17}$$

Although our results have been derived for the example of the stadium billiard, they are more generally valid. They apply to all billiard systems in which families of neutral periodic orbits exist, in which every primitive member has the same length and a phase factor $\exp\{i\pi\nu/2\} = 1$. a then corresponds to half of the primitive length of a periodic orbit, and b is the geometrical width of the family.

2.2. *The contributions of the unstable periodic orbits*

The oscillatory contributions of the 'bouncing ball orbits' are similar in form to the contributions of the isolated unstable periodic orbits which are given by

$$\hat{d}_\gamma^{\mathrm{osc}}(k) = \frac{l_\gamma}{\pi} \sum_{n=1}^{\infty} \frac{\cos\{nkl_\gamma - n\pi\nu_\gamma/2\}}{\sqrt{|2 - \mathrm{Tr}\, M_\gamma^n|}} \tag{18}$$

where l_γ and M_γ are the length and the monodromy matrix of the orbit γ, respectively, and n is the number of repetitions. ν_γ is equal to the maximum number of conjugate points along the orbit, plus twice the number of its reflections on the boundary. The contributions of the family of neutral trajectories are in leading order of $\hbar$ larger by a factor of $\hbar^{-1/2}$. However, the number of unstable orbits is exponentially proliferating and for that reason their contribution cannot be neglected [8].

2.3. *The edge contributions for the family of neutral periodic orbits*

In the derivation above edge effects have been neglected. In the vicinity of the two limiting orbits of the family of neutral periodic orbits there are closed almost periodic orbits, i.e. their final momentum is almost parallel to the initial momentum. These orbits give a contribution to the level density, which is of the same order in $\hbar$ as the contribution of an unstable periodic orbit. The leading term of this contribution can be obtained analogously to the derivation of the contribution of an unstable periodic orbit to Gutzwiller's trace formula. The actions of the closed orbits in the vicinity of the two limiting periodic orbits are expanded up to second order, and the trace of the Green function is evaluated by a stationary phase approximation.

We first consider the limiting neutral periodic orbit, which is reflected at the intersection between the straight line and the quarter circle (see figure 5). For $n = 1$, i.e. for one traversal of the limiting periodic orbit, there exist only neighbouring closed orbits which stay in the quarter circle. Their contribution is given by

$$\hat{d}_{\mathrm{e1}}^{\mathrm{osc}}(k) = \frac{l_{\mathrm{e1}}}{2\pi} \frac{\cos\{kl_{\mathrm{e1}} - \pi\nu_{\mathrm{e1}}/2\}}{\sqrt{|2 - \mathrm{Tr}\, M_{\mathrm{e1}}|}} \tag{19}$$

where $l_{\mathrm{e1}} = 2a$, $\mathrm{Tr}\, M_{\mathrm{e1}} = -2$ and $\nu_{\mathrm{e1}} = 5$ are the values for a periodic orbit along the symmetry line of a semicircle. In contrast to the contribution of an unstable periodic orbit there is an additional factor 1/2 since one integrates only over the closed orbits on one side of the periodic orbit. The contributions from multiple traversals of the limiting periodic orbit invlove the integration over several families of neighbouring closed orbits. They are not treated here.

The derivation above did not take into account, that the curvature of the boundary is discontinuous at the intersection between the straight line and the quarter circle, and for that reason the semiclassical Green function is also discontinuous. A more accurate analysis can be carried out by adding corrections to the semiclassical Green function due to diffraction effects at the intersection point. A further analysis, however, shows that after taking the trace of the Green function the diffraction effects are cancelled in leading order of $\hbar$. For that reason, the leading term of the edge contribution is given correctly by (19) for the case $n = 1$.

In the case of the second limiting periodic orbit which runs along the left straight section of the boundary of length a (see figure 5), the neighbouring closed orbits have the property, that they are reflected n times on the upper, n times on the lower and one time on the left part of the boundary. Their contribution is given by

$$\hat{d}_{e2}^{osc}(k) = \frac{l_{e2}}{2\pi} \sum_{n=1}^{\infty} \frac{\cos\{nkl_{e2} - n\pi\nu_{e2}/2 - \pi\}}{\sqrt{|2 + \mathrm{Tr}\, M_{e2}^n|}} \tag{20}$$

where $l_{e2} = 2a$, $\mathrm{Tr}\, M_{e2} = 2$ and $\nu_{e2} = 4$ are the values for one of the bouncing ball orbits. Again there is an additional factor of 1/2 in comparison to (18), since one integrates only over closed orbits on one side of the limiting periodic orbit. A discussion of the symmetry origin of this factor 1/2 can be found in [14–16]. Furthermore, there is an additional $(-\pi)$ in the argument of the cosine, and the trace of the monodromy matrix contributes with a different sign. This is due to the reflection on the left side of the boundary.

2.4. Further edge contributions

There are also contributions from closed almost periodic orbits in the vicinity of the unstable periodic orbit which runs along the lower straight section of the boundary of length $(a+b)$. Again the derivation of this contribution can be done in analogy to the derivation of the contribution of an unstable periodic orbit for Gutzwillers trace formula. For every repetition number n one has to distinguish, however, between two families of closed orbits, one with an odd number of reflections on the lower part of the boundary and the other with an even number of such reflections. One obtains

$$\hat{d}_{e3}^{osc}(k) = \frac{l_{e3}}{2\pi} \sum_{n=1}^{\infty} \left[\frac{\cos\{nkl_{e3} - n\pi\nu_{e3}/2\}}{\sqrt{|2 - \mathrm{Tr}\, M_{e3}^n|}} + \frac{\cos\{nkl_{e3} - n\pi\nu_{e3}/2 - \pi\}}{\sqrt{|2 + \mathrm{Tr}\, M_{e3}^n|}} \right] \tag{21}$$

where $l_{e3} = 2(a+b)$, $\mathrm{Tr}\, M_{e3} = -2 - 4b/a$ and $\nu_{e3} = 5$. Again there is an additional factor of 1/2. The first and the second term in the sum are obtained from closed orbits with an even and an odd number of reflections on the lower part of the boundary, respectively.

2.5. The contribution of the 'whispering gallery orbits'

The situation is more complicated in the case of the upper part of the boundary which consists of a straight and a circular section. There is a periodic orbit of length $l_\gamma = 2b + \pi a \approx 135$ which runs along this part of the boundary. It can be considered as the limit of a family of an infinite number of periodic orbits ('whispering gallery orbits'). These orbits have the property that they are reflected consecutively n times at the circular part of the boundary. After these n reflections they are reflected back either at a straight section of the boundary or at a corner and retrace themselves. For every $n \in \mathbb{N}$ there are four orbits corresponding to the four cases that the two 'endpoints' of the self-retracing orbits can be either at a straight section of the boundary or at a corner. There have been doubts if the semiclassical approximation is valid for these periodic orbits, since the distance between neighbouring periodic orbits becomes arbitrarily small as the number n increases. There is, however, a high cancellation between the contributions of pairs of periodic orbits, such that effectively only a finite number of periodic orbits contribute. This will be demonstrated by numerical results in the next chapter. For that reason, we treat the 'whispering gallery orbits' as ordinary periodic orbits, i.e. their contribution to the level density is assumed to be given by (18).

Summing up all contributions to the semiclassical level density one has

$$\hat{d}(k) \approx \langle \hat{d}(k) \rangle + \hat{d}_{\mathrm{b}}^{\mathrm{osc}}(k) + \hat{d}_{\mathrm{e}}^{\mathrm{osc}}(k) + \sum_{\gamma} \hat{d}_{\gamma}^{\mathrm{osc}}(k) \tag{22}$$

where $\hat{d}_{\mathrm{e}}^{\mathrm{osc}}(k)$ is the sum of the three different edge contributions.

3. Fourier analysis of the spectrum

The contributions of the periodic orbits to the level density can be extracted from the spectrum by a Fourier analysis. Since in an experimental situation one has knowledge only of a finite part of the spectrum we consider an integral which is cut off at a maximal wavenumber k_{max}

$$\hat{D}(x, k_{\mathrm{max}}) = \frac{1}{k_{\mathrm{max}}} \int_0^{k_{\mathrm{max}}} \mathrm{d}k\, \cos(kx)[\hat{d}(k) - \langle \hat{d}(k) \rangle]. \tag{23}$$

This function has peaks (or zeros) at lengths of periodic orbits. At this point the semiclassical approximation of the level density (22) is inserted into (23) in order to obtain the contribution of single periodic orbits. In general the sum over periodic orbits in (22) is not convergent. For that reason one has to consider a smoothed level density, for example a level density which is obtained by a Gaussian smoothing of (22) [17]. This also corresponds more closely to the experimental situation in which one always has a level density with peaks of finite widths. However, the smoothing can be chosen arbitrarily small, and since we will consider only the contributions of a finite number of the shortest periodic orbits we will neglect it in the following. In this way we obtain peaks corresponding to 'bouncing ball orbits' of the form

$$\hat{D}_{\mathrm{b}}(x, k_{\mathrm{max}}) = \frac{b}{2\pi} \sqrt{\frac{a k_{\mathrm{max}}}{2\pi}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \{ f[k_{\mathrm{max}}(2an - x)] + f[k_{\mathrm{max}}(2an + x)] \} \tag{24}$$

where $f(z)$ is a line-shape function which can be expressed in terms of Fresnel functions [11]

$$f(z) = -\sqrt{\frac{\pi}{2|z|}} \frac{S(|z|^{1/2})}{|z|} + \frac{\sin(z)}{z} + \sqrt{\frac{\pi}{2|z|}} \frac{C(|z|^{1/2})}{z} - \frac{\cos(z)}{z} \tag{25}$$

and is plotted in figure 1.

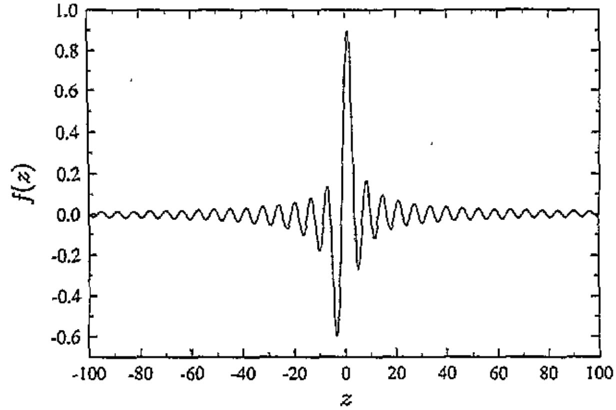

Figure 1. The line-shape function $f(z)$ as defined in (25).

The contribution of an isolated unstable orbit is given by

$$\hat{D}_\gamma(x,k_{\max}) = \frac{l_\gamma}{2\pi}\sum_{n=1}^{\infty}\frac{1}{\sqrt{|2-\operatorname{Tr} M_\gamma^n|}}\left\{\frac{\sin[k_{\max}(x-nl_\gamma)+\pi n\nu_\gamma/2]-\sin[\pi n\nu_\gamma/2]}{k_{\max}(x-nl_\gamma)}\right.$$
$$\left.+\frac{\sin[k_{\max}(x+nl_\gamma)-\pi n\nu_\gamma/2]+\sin[\pi n\nu_\gamma/2]}{k_{\max}(x+nl_\gamma)}\right\}. \tag{26}$$

The edge contributions can be obtained from (26) straightforwardly by taking into account the differences between $\hat{d}_\gamma^{\rm osc}(k)$ and $\hat{d}_{\rm e1}^{\rm osc}(k)$, $\hat{d}_{\rm e2}^{\rm osc}(k)$ and $\hat{d}_{\rm e3}^{\rm osc}(k)$, respectively.

One main difference between the peaks of unstable orbits and the family of neutral orbits is that the height of the peaks of the family of orbits depends on the cutoff $k_{\max}$ whereas the peaks of unstable orbits do not. In this way the peaks at the neutral orbits can be recognized by varying $k_{\max}$.

Note that the function (26) yields a positive or negative peak at the position of the length of the nth traversal of a primitive periodic orbit, if the number $n\nu_\gamma$ is even. In the case where this number is odd one obtains a zero at the position of the length, which lies between a positive and a negative peak.

We present some numerical results for the finite Fourier transform $\hat{D}(x,k_{\max})$. They have been evaluated using the first 1060 energy eigenvalues of a spectrum. These energies were determined in an experiment which measured the eigenmodes of a superconducting microwave resonator in the shape of a quarter of a stadium billiard with $a = 20$ and $b = 36$ [4]. (We use dimensionless units in which $\hbar = 2m = 1$.) The experimental results are then compared with semiclassical results using periodic orbits. Differences between both results are due to experimental limitations and/or errors in the semiclassical approximation.

Figure 2 shows the function $\hat{D}(x,k_{\max})$ in the vicinity of $x = 40$, which corresponds to the first traversal of the 'bouncing ball orbits'. In this region of x the contributions of the unstable periodic orbits can be neglected, so that $\hat{D}(x,k_{\max})$ is mainly given by the contribution of the family of neutral orbits. There is a very good agreement between $\hat{D}(x,k_{\max})$ and $\hat{D}_{\rm b}(x,k_{\max})$. Figure 3 shows the difference between these two functions. This difference is compared with the edge correction $\hat{D}_{\rm e1}(x,k_{\max}) + \hat{D}_{\rm e2}(x,k_{\max})$. Here the agreement between both curves is not very good. However, since the edge correction is a small effect in comparison to the main contribution of the bouncing ball orbits, its

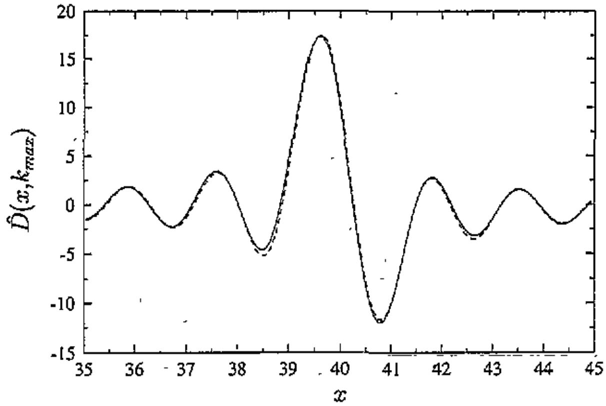

Figure 2. The function $\hat{D}(x, k_{\max})$ (full curve) in comparison with $\hat{D}_b(x, k_{\max})$ (broken curve).

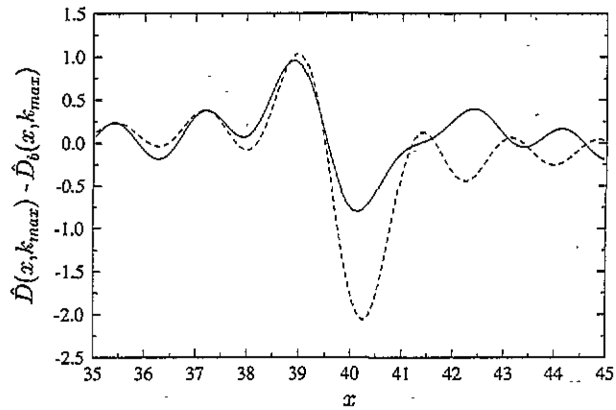

Figure 3. The difference between the two curves in figure 2 $\hat{D}(x, k_{\max}) - \hat{D}_b(x, k_{\max})$ (full curve) in comparison with the edge contributions $\hat{D}_{e1}(x, k_{\max}) + \hat{D}_{e2}(x, k_{\max})$ (broken curve).

determination from the experimental data is very sensitive to the exact geometrical data of the microwave cavity. For the evaluation of figures 2 and 3 we already used a corrected value of the length $a = 19.96$, which was obtained from the position of the first peaks at the lengths of the bouncing ball orbits and their repetitions. There might be some other geometrical inaccuracies, for example a slight imperfection of the cavity in the transition region between the straight and the circular part of the boundary such that the radius of curvature does not change abruptly but smoothly [4].

In figure 4 $\hat{D}(x, k_{\max})$ is shown in a region where many unstable periodic orbits contribute and there are also multiple traversals of the family of neutral orbits. All periodic orbits with length below $x = 200$ have been determined. There is a good overall agreement between the experimental and theoretical results. At $x \approx 142$ there is some deviation between both curves, which is not fully understood. It might be connected with an unstable periodic orbit which has two reflection points in the vicinity of the intersection of the circular and the straight section of the boundary (see figure 5). This orbit is sensitive to the exact geometrical shape of the cavity in this region and thus might be affected by a geometrical imperfection as mentioned above.

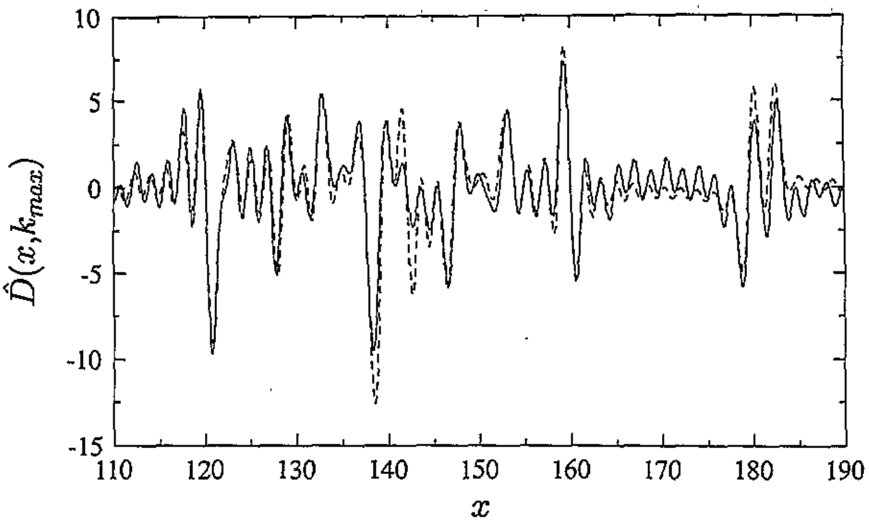

Figure 4. The function $\hat{D}(x, k_{\max})$ (full curve) in comparison with its semiclassical approximation (broken curve).

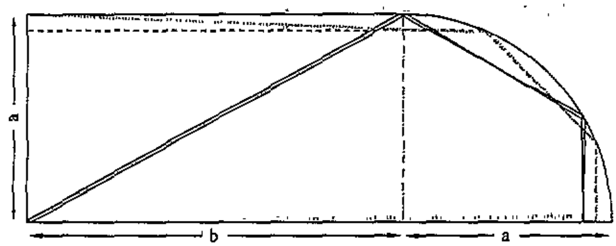

Figure 5. Two of the 'whispering gallery orbits' (broken and dotted curves), and a non-self-retracing orbit (full curve), which has two reflection points close to the intersection between the straight segment and the circular segment of the boundary. The chain curve shows the last 'bouncing ball orbit'.

As was discussed in the last section there is an infinite number of periodic orbits with length below $x = 200$, since there is an accumulation point of periodic orbits at $x = 2b + \pi a \approx 135$ ('whispering gallery orbits'). However, the experimental length spectrum does not reveal any special structure for $x \approx 135$, and the agreement between semiclassical theory and experiment is rather good in this domain. A possible explanation for this is the fact that there exists a high cancellation between the contributions of pairs of periodic orbits, such that effectively only a finite number of periodic orbits contribute. For every orbit which has a reflection point on the left straight vertical section of the billiard boundary, there exists another very similar orbit which has a reflection point in the upper left corner of the billiard boundary (see figure 5). Both orbits have almost the same length and instability exponent, but the number ν_y differs by two. Thus they give an almost identical contribution to the periodic-orbit sum but with a different sign. This is shown in figure 6 for two pairs of orbits with $n = 2$ and $n = 3$. The effectiveness of the cancellation increases very rapidly with n and for $n = 3$ it is almost complete.

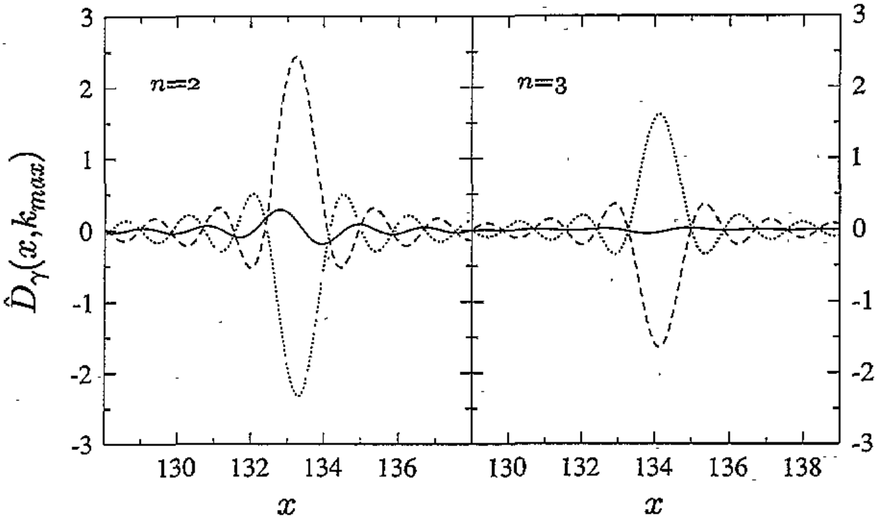

Figure 6. Left: The contributions of the two 'whispering gallery orbits' of figure 5 to $\hat{D}(x, k_{\max})$ (broken and dotted curves). The full curve shows the sum of the two contributions, which almost cancel each other. Right: The contributions of two 'whispering gallery orbits', which have three reflection points on the circular section of the boundary.

4. Spectral statistics

In this section we discuss the influence of the presence of the family of neutral periodic orbits to energy statistics which are bilinear in the level density. For that purpose we consider a two-point correlation function of the oscillatory part of the level density

$$k(\varepsilon) = \langle d^{\rm osc}(E+\varepsilon/2) d^{\rm osc}(E-\varepsilon/2)\rangle / \langle d(E)\rangle \; . \tag{27}$$

Equation (27) has to be evaluated in the semiclassical regime where E is large. $\langle\ \rangle$ denotes an energy averaging over an energy interval which is small in comparison to E, but contains a large number of energy levels. $d^{\rm osc}(E)$ semiclassically consists of the contributions of the two kind of periodic orbits

$$\begin{aligned} k(\varepsilon) = \big\langle & [d_{\rm b}^{\rm osc}(E+\varepsilon/2) + d_{\rm r}^{\rm osc}(E+\varepsilon/2)] \\ & \times [d_{\rm b}^{\rm osc}(E-\varepsilon/2) + d_{\rm r}^{\rm osc}(E-\varepsilon/2)]\big\rangle / \langle d(E)\rangle \; . \end{aligned} \tag{28}$$

(the edge contributions are included in $d_{\rm r}^{\rm osc}(E)$). At this point the semiclassical approximation for the level density is inserted. One obtains a double sum over periodic orbits, which contains energy-dependent oscillatory terms of the form $\cos(S_i - S_j)$ and $\cos(S_i + S_j)$, respectively, where S_i and S_j are the classical actions of the indifferent or unstable periodic trajectories. Because of the energy averaging over a semiclassically large interval ΔE most of the non-diagonal terms in the double sum can be neglected. Only terms with very small action differences will not be washed out by the energy averaging. We assume here that due to this energy averaging the interference terms between the contributions of the neutral trajectories and the contributions of the unstable trajectories can be neglected in the semiclassical regime. Then

$$\begin{aligned} k(\varepsilon) &= \langle d_{\rm b}^{\rm osc}(E+\varepsilon/2) d_{\rm b}^{\rm osc}(E-\varepsilon/2)\rangle / \langle d(E)\rangle + \langle d_{\rm r}^{\rm osc}(E+\varepsilon/2) d_{\rm r}^{\rm osc}(E-\varepsilon/2)\rangle / \langle d(E)\rangle \\ &=: k_{\rm b}(\varepsilon) + k_{\rm r}(\varepsilon) \; . \end{aligned} \tag{29}$$

$k_r(\varepsilon)$ contains only contributions from the unstable periodic orbits. According to the theory of Berry [5] $k_r(\varepsilon)$ (after unfolding) agrees with the correlation function of the Gaussian orthogonal ensemble (GOE) for $\varepsilon \ll \varepsilon_{max}$, where $\varepsilon_{max} = 2\pi\hbar/T_{min}$ and T_{min} is the period of the shortest unstable periodic orbit. For $\varepsilon \gg \varepsilon_{max}$ $k_r(\varepsilon)$ deviates from the GOE expectation due to the presence of a shortest periodic orbit.

$k_b(\varepsilon)$ is evaluated by inserting the oscillatory part of $d_b(E)$ from equations (9) and (11)

$$\begin{aligned} k_b(\varepsilon) &= \left\langle \frac{m^2ab^2}{2\pi^3\hbar^4\langle d(E)\rangle} \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} \frac{1}{\sqrt{n_1 n_2 k_+ k_-}} [\cos\{2an_1k_+ + 2an_2k_- - \pi/2\} \right. \\ &\quad \left. + \cos\{2a(n_1k_+ - n_2k_-)\}] \right\rangle \\ &\approx \left\langle \frac{m^2ab^2}{2\pi^3\hbar^4\langle d(E)\rangle\sqrt{k_+k_-}} \sum_{n=1}^{\infty} \frac{1}{n} \cos\{2an(k_+ - k_-)\} \right\rangle \\ &\approx \frac{m^2ab^2}{2\pi^3\hbar^4 k\langle d(E)\rangle} \sum_{n=1}^{\infty} \frac{1}{n} \cos\left\{\frac{2man\varepsilon}{\hbar^2 k}\right\} \\ &= \frac{m^2ab^2}{4\pi^3\hbar^4 k\langle d(E)\rangle} \log\left\{\frac{1}{2}\left(1-\cos\frac{2ma\varepsilon}{\hbar^2 k}\right)^{-1}\right\} \end{aligned} \tag{30}$$

where $k_+ = \sqrt{2m(E+\varepsilon/2)}/\hbar$ and $k_- = \sqrt{2m(E-\varepsilon/2)}/\hbar$. We made use of the fact that due to the energy averaging only the diagonal terms of the double sum give a significant contribution to $k_b(\varepsilon)$, and we consider values $\varepsilon \ll E$.

The number variance $\Sigma^2(L)$ is defined as the local variance of the number of energy levels in an energy interval, which has the size of L mean level spacings. It is expressed in terms of the correlator as

$$\Sigma^2(L) = 4\int_0^{L/2} du \int_0^{2u/\langle d(E)\rangle} dv\, k(v) . \tag{31}$$

Using (29) the semiclassical approximation to $\Sigma^2(L)$ can be divided into a part with contributions from the neutral periodic orbits and a part with contributions from the unstable periodic orbits. With (30) one obtains the 'bouncing ball orbits' contribution to $\Sigma^2(L)$ as

$$\Sigma_b^2(L) = -\frac{kb^2}{4\pi^3 a}\sum_{n=1}^{\infty}\frac{1}{n^3}\left[\cos\left\{\frac{2manL}{\hbar^2 k\langle d(E)\rangle}\right\} - 1\right]. \tag{32}$$

Another statistic is the spectral form factor $K(\tau)$ which is defined as

$$K(\tau) = \int_{-\infty}^{\infty} d\varepsilon\, \exp\{i\varepsilon T/\hbar\} k(\varepsilon) . \tag{33}$$

where $\tau = T/(2\pi\hbar\langle d(E)\rangle)$ is time in units of the Heisenberg time. For very large energies we can use (30) in order to obtain the contributions of the neutral periodic orbits to $K(\tau)$ as

$$K_b(\tau) = \frac{m^2ab^2}{4\pi^3\hbar^4 k\langle d(E)\rangle^2}\sum_{n=1}^{\infty}\frac{1}{n}\left[\delta\left(\tau + \frac{man}{\pi\hbar^2 k\langle d(E)\rangle}\right) + \delta\left(\tau - \frac{man}{\pi\hbar^2 k\langle d(E)\rangle}\right)\right] \tag{34}$$

i.e. $K_b(\tau)$ consists of a series of delta peaks at times which correspond to traversals of orbits of lengths $2an$.

$K_b(\tau)$ and $\Sigma_b^2(L)$ are related through the equation

$$\Sigma^2(L) = \frac{2}{\pi^2}\int_0^\infty d\tau \frac{K(\tau)}{\tau^2}\sin^2(\pi L\tau)\,. \tag{35}$$

Finally the contribution to the spectral rigidity is obtained as

$$\begin{aligned}\Delta_3^b(L) &= \frac{2}{L^4}\int_0^L dr\,(L^3 - 2L^2r + r^3)\Sigma^2(r)\\ &= \frac{b^2k}{2\pi^3aL^4}\sum_{n=1}^\infty \frac{1}{n^3}\left\{\frac{L^4}{4} + \frac{6L}{z^3n^3}\sin(znL) - \frac{L^2}{z^2n^2}[\cos(znL)+2]\right.\\ &\quad\left. + \frac{6}{z^4n^4}[\cos(znL) - 1]\right\}\end{aligned} \tag{36}$$

where $z = 2ma/(\hbar^2k\langle d(E)\rangle)$. This result is in good agreement with the experimental result of Gräf *et al* [4].

Figure 7 shows an evaluation of $\Delta_3^b(L)$ for a ratio $b/a = 1.8$ at an energy which satisfies $\langle N(E)\rangle = 500$.

In the limit $L \to \infty$, $\Delta_3^b(L)$ *saturates*

$$\lim_{L\to\infty}\Delta_3^b(L) = \frac{b^2k\zeta(3)}{8\pi^3 a} \tag{37}$$

this result has been previously obtained by Berry [5].

In summary, we calculated the semiclassical contribution of a family of neutral periodic orbits to the spectral density of a strongly chaotic system. These families give stronger contributions to the level density than single unstable orbits, on the other hand the unstable orbits are far more numerous due to their exponential proliferation in a chaotic system. We have shown that the combined effect of the neutral orbits and the unstable ones gives a fair description of the experimental spectrum when used within the semiclassical approximation. The contributions of the neutral orbits lead to a deviation of the statistical distribution of energy levels from the distribution according to random matrix theory. This is a deviation in addition to the deviation which is due to the presence of a shortest unstable periodic orbit. The deviation is calculated for several statistics which are bilinear in the level density. The theoretical results agree well with experimental results for the stadium billiard.

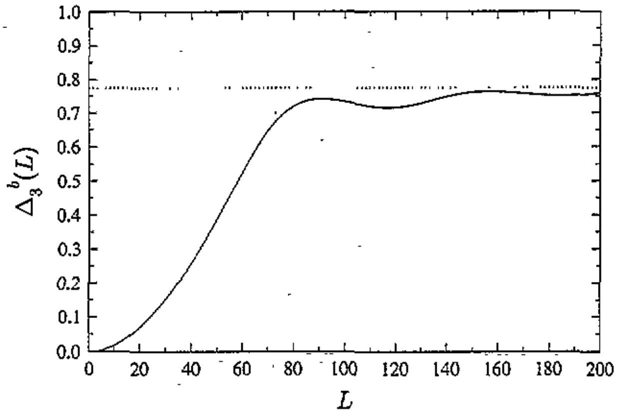

Figure 7. The contribution $\Delta_3^b(L)$ of the family of neutral periodic orbits to the spectral rigidity.

Acknowledgments

We are grateful to Professor Achim Richter for the kind provision of the eigenfrequencies of the stadium-shaped microwave cavity. This research was supported in part by grants from the US–Israel Binational Science Foundation (BSF) and the Basic Research Foundation of the Israel Academy of Sciences. MS would like to thank MINERVA for financial support.

References

[1] Bunimovich L A 1979 *Commun. Math. Phys.* **65** 295
[2] Bunimovich L A 1991 *Chaos* **1** 187
[3] McDonald S W and Kaufman A N 1979 *Phys. Rev. Lett.* **42** 1189
[4] Gräf S W, Harney H L, Lengeler H, Lewenkopf C W, Rangacharyulu C, Richter A, Schardt P and Weidenmüller H A 1992 *Phys. Rev. Lett.* **69** 1296
[5] Berry M V 1985 *Proc. R. Soc.* A **400** 229
[6] Gutzwiller M C 1971 *J. Math. Phys.* **12** 343
[7] Gutzwiller M C 1990 *Chaos in Classical and Quantum Mechanics* (New York: Springer)
[8] Berry M V 1981 *Ann. Phys., NY* **131** 163
[9] Creagh S C and Littlejohn R G 1991 *Phys. Rev.* A **44** 836
[10] Berry M V and Tabor M 1977 *J. Phys. A: Math. Gen.* **10** 371
[11] Gradshteyn I S and Ryzhik I M 1965 *Table of Integrals, Series, and Products* (New York: Academic)
[12] Baltes H P and Hilf E R 1976 *Spectra of Finite Systems* (Mannheim: Bibliographisches Institut)
[13] Keating J P and Berry M V 1987 *J. Phys. A: Math. Gen.* **20** L1139
[14] Hönig A and Wintgen D 1989 *Phys. Rev.* A **39** 5642
[15] Sieber M 1991 *PhD Thesis* University of Hamburg
[16] Lauritzen B 1991 *Phys. Rev.* A **43** 603
[17] Aurich R, Sieber M and Steiner F 1988 *Phys. Rev. Lett.* **61** 483